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Application of material forces to hyperelastostatic fracture mechanics. I. Continuum mechanical setting

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Dedicated to the occasion of the 70th birthday of Professor Dr. rer. nat. H.G. Hahn

Abstract

The concern of this work is a consequent exploitation of the notion of material forces for the application within hyperelastostatic fracture mechanics. Contrary to physical forces, material forces act on the material manifold, thus essentially representing the tendency of defects like cracks or inclusions to move relative to the ambient material. Based on the formulation of the appropriate quasi-static balance laws in the material space we aim at a fresh look onto classical aspects of hyperelastostatic fracture mechanics. Operating throughout within the geometrically nonlinear setting we emphasize on the one hand the duality of the direct and the inverse motion description and on the other hand we re-establish the classical path integrals from elementary equilibrium considerations in the material space. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Based on the reformulation of common physical balance equations on the material manifold, essentially obtained by appropriate pull-back operations, many aspects of hyperelastostatic fracture and defect mechanics can be clarified in a unified framework. In particular, well-known path integrals together with the discussion of their path independence are re-established from elementary equilibrium type considerations. To this end, the pertinent balance equations together with the corresponding fluxes pertaining to the direct and the inverse motion description have to be considered in parallel. This viewpoint embraces a number of classical continuum mechanical aspects and opens the door for new computational strategies, which will be treated separately in the second part of this work.

The concept of an energy-momentum tensor is in the heart of the present treatise. It has been introduced by Eshelby (1951, 1956, 1970, 1975) in the early 1950s to study defects in elastic continua. To honor

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Eshelby, we will therefore address the energy-momentum tensor in the sequel as the Eshelby stress, moreover for the branch of continuum mechanics dealing with singularities and inhomogeneities. Maugin (1993) recently coined the terminology Eshelbian mechanics. The forces associated with the Eshelby stresses are commonly denoted configurational, quasi-Newton or rather material forces, (e.g. Rogula, 1977), whereby we will adopt the latter terminology in the sequel.

Conservation laws in hyperelasticity resulting from Noether's theorem were studied e.g. by Günther (1962), Knowles and Sternberg (1972), Fletcher (1976), Rogula (1977) and Olver (1984). Clearly, conservation laws are closely related to solenoidal, i.e. divergence free fields and consequently to path independent integrals.

Consequently, as an early outcome of these investigations three path independent integrals, which are commonly referred to as *J*- *L*- and *M*-integral, have been discovered. Thereby, it turns out that the Eshelby stress is the main ingredient of these integrals. The application of the path independent *J*-integral to fracture mechanics was earlier proposed in the celebrated publication by Rice (1968), the interpretation of the *J*- *L*- and *M*-integrals as being energetically conjugated to translations, rotations and scale changes of defects within elastic material goes back to Budiansky and Rice (1973). Applications of a variant of the *M*-integral in finite elasticity are considered by Green (1973).

The inverse motion point of view is intimately related to the aforementioned developments. Thereby, inverse motion problem and its duality with the direct motion problem, in particular the duality of the Cauchy stress and the Eshelby stress, has been studied by Shield (1967), Chadwick (1975), Ogden (1975) and Golebiewska-Herrmann (1981), among others.

In his reflections on the general theory of energy-momentum tensors in elastostatics Hill (1986) investigated the implications of the energy-momentum tensor on the changes in potential energy of a hyperelastic continuum due to arbitrary variations of material inhomogeneities.

The application of the notion of material forces to classical strength of materials problems within the elementary beam theory has been considered e.g. by Kienzler (1986) and Kienzler and Herrmann (1986a,b). More recently, e.g. Stumpf and Le (1990) and Maugin and Trimarco (1992) developed the variational setting of Eshelbian mechanics with particular application to brittle fracture. As an extension, Maugin (1994) further investigated the relation between material forces and the *J*-integral for dynamical fracture in elastic and electromagneto-elastic continua. A comprehensive treatment of material inhomogeneities in elasticity is presented in the recent monograph by Maugin (1993) which promotes as well the concept of material forces.

It is the objective of this work to shed an alternative light on different aspects of hyperelastostatic fracture mechanics based on the consequent use of the concept of material forces. Thereby, even if many hyperelastic fracture mechanics problems are characterized as being brittle and may therefore be solved within the geometrically linear setting, *it is not by snobbery or some mathematical pedantry*, *but for its heuristic value that we present all developments first in the frame of finite strains*, as Maugin (1995) clearly pointed out.

To this end, Part I is organized as follows: In Section 2, the kinematics of the direct and the inverse motion description are briefly reiterated. In Section 3, we contrast the common physical viewpoint, resting on the the concept of physical surface and volume forces in the sense of Newton, with the complementary material viewpoint, based on the concept of material surface and volume forces in the sense of Eshelby. Then, Section 4 develops the pertinent hyperelastic stress measures and the quasi-static balance equations for the physical and the material viewpoint. These balance equations are subsequently related to an appropriate variational setting in Section 5. Finally, Section 6 reflects on the motivation of the classical path integrals based on the aforementioned developments. We finally close with conclusions in Section 7.

The computational setting emanating from the advocated view on the application of material forces to hyperelastostatic fracture mechanics is highlighted separately in the forthcoming Part II of this work.

2. Kinematics

To set the stage, we briefly reiterate the geometrically nonlinear kinematics of the direct and the inverse motion description.

2.1. Direct motion description

In the direct motion description the placement $\boldsymbol{x}$ of a material particle in the current configuration $\mathscr{B}$ is described by the nonlinear direct deformation map $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X})$ in terms of the placement $\boldsymbol{X}$ of the same material particle in the reference configuration $\mathscr{B}_0$ (Fig. 1). The direct deformation gradient, i.e. the linear tangent map associated with the direct deformation, together with its determinant are denoted by $\boldsymbol{F} = \nabla_X \boldsymbol{\varphi}$ and $J = \det \boldsymbol{F}$, respectively. Typical strain measures are then defined by the right and left Cauchy–Green strain tensors $\boldsymbol{C} = \boldsymbol{F}^t \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^t$, respectively, of the direct motion.

For later use, we note that for a direct deformation relating compatible configurations $\mathscr{B}_0$ and $\mathscr{B} = \boldsymbol{\varphi}(\mathscr{B}_0)$, the integrability condition for the direct deformation gradient reads $\mathrm{Rot}\boldsymbol{F} = \boldsymbol{0}$ and renders the identity $\nabla_X \boldsymbol{F}^t : [\bullet] = [\bullet] : \nabla_X \boldsymbol{F}$ for any second order two-point tensor $[\bullet]$ mapping between the tangent spaces to $\mathscr{B}_0$ and $\mathscr{B}$.

Moreover, we define the physical variation of a quantity $[\bullet]$ at fixed reference placement $\boldsymbol{X}$ as $\delta_X[\bullet] = d_\epsilon[[\bullet](\boldsymbol{\varphi} + \epsilon\delta\boldsymbol{\varphi}; \boldsymbol{X})]_{\epsilon=0}$ with the obvious commutation rule $\nabla_X \delta_X[\bullet] = \delta_X \nabla_X[\bullet]$. As an example, we note that $\delta_X \boldsymbol{F} = \nabla_X \delta\boldsymbol{\varphi}$ rendering the physical variation of $\boldsymbol{F}^{-1}$ as $\delta_X \boldsymbol{F}^{-1} = -\boldsymbol{F}^{-1} \cdot \nabla_x \delta\boldsymbol{\varphi}$. Likewise, we obtain the physical variation for the inverse of $\boldsymbol{\varphi}$ as $\delta_X \boldsymbol{\varphi}^{-1} = -\boldsymbol{F}^{-1} \cdot \delta\boldsymbol{\varphi}$, Maugin and Trimarco (1992).

2.2. Inverse motion description

Accordingly, in the inverse motion description, the placement $\boldsymbol{X}$ of a material particle in the reference configuration $\mathscr{B}_0$ is described by the nonlinear inverse deformation map $\boldsymbol{X} = \boldsymbol{\phi}(\boldsymbol{x})$ in terms of the placement $\boldsymbol{x}$ of the same material particle in the current configuration $\mathscr{B}$ (Fig. 2). The inverse deformation gradient, i.e. the linear tangent map associated to the inverse deformation, together with its determinant are denoted by $\boldsymbol{f} = \nabla_x \boldsymbol{\phi}$ and $j = \det \boldsymbol{f}$, respectively. Typical strain measures are then defined by the left and right Cauchy–Green strain tensors $\boldsymbol{c} = \boldsymbol{f}^t \cdot \boldsymbol{f}$ and $\boldsymbol{B} = \boldsymbol{f} \cdot \boldsymbol{f}^t$, respectively, of the inverse motion.

Again, for later use, we note that for an inverse deformation relating compatible configurations $\mathscr{B}$ and $\mathscr{B}_0 = \boldsymbol{\phi}(\mathscr{B})$, the integrability condition for the inverse deformation gradient reads $\mathrm{rot}\boldsymbol{f} = \boldsymbol{0}$ and renders the identity $\nabla_x \boldsymbol{f}^t : [\bullet] = [\bullet] : \nabla_x \boldsymbol{f}$ for any second order two-point tensor $[\bullet]$ mapping between the tangent spaces to $\mathscr{B}$ and $\mathscr{B}_0$.

Moreover, we define the material variation of a quantity $[\bullet]$ at fixed current placement $\boldsymbol{x}$ as $\delta_x[\bullet] = d_\epsilon[[\bullet](\boldsymbol{\phi} + \epsilon\delta\boldsymbol{\phi}; \boldsymbol{x})]_{\epsilon=0}$ with the obvious commutation rule $\nabla_x \delta_x[\bullet] = \delta_x \nabla_x[\bullet]$. As an example, we note

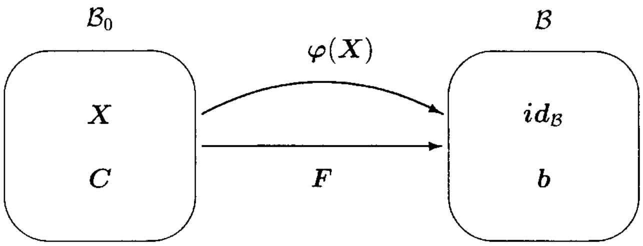

Fig. 1. Kinematics of the direct motion description: direct deformation map and corresponding deformation gradient together with left and right Cauchy–Green strain tensors (reference placement and identity map).

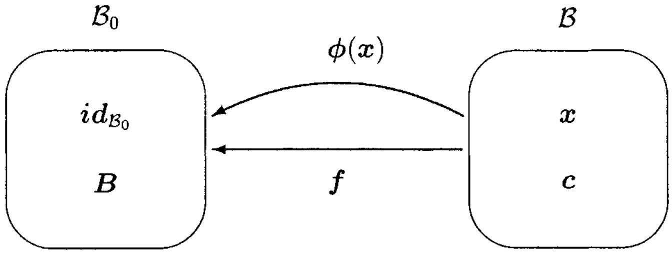

Fig. 2. Kinematics of the inverse motion description: inverse deformation map and corresponding deformation gradient together with left and right Cauchy–Green strain tensors (current placement and identity map).

that $\delta_x \boldsymbol{f} = \nabla_x \delta\boldsymbol{\phi}$, rendering the material variation of $\boldsymbol{f}^{-1}$ as $\delta_x \boldsymbol{f}^{-1} = -\boldsymbol{f}^{-1} \cdot \nabla_X \delta\boldsymbol{\phi}$. Likewise, we obtain the material variation for the inverse of $\boldsymbol{\phi}$ as $\delta_x \boldsymbol{\phi}^{-1} = -\boldsymbol{f}^{-1} \cdot \delta\boldsymbol{\phi}$, see Maugin and Trimarco (1992).

Remark 2.1. *Please note that the direct and the inverse motion descriptions are related by the identity maps* $\boldsymbol{id}_{\mathscr{B}_0} = \boldsymbol{\phi} \circ \boldsymbol{\varphi}(\boldsymbol{X})$ *and* $\boldsymbol{id}_{\mathscr{B}} = \boldsymbol{\varphi} \circ \boldsymbol{\phi}(\boldsymbol{x})$ *together with the inverse relations* $\boldsymbol{F}^{-1} = \boldsymbol{f} \circ \boldsymbol{\varphi}(\boldsymbol{X})$ *and* $\boldsymbol{f}^{-1} = \boldsymbol{F} \circ \boldsymbol{\phi}(\boldsymbol{x})$, *whereby* $\circ$ *denotes composition.*

3. Physical versus material viewpoint

In this section, we emphasize the formal duality of physical and material forces acting on arbitrary subdomains of a body and the corresponding quasi-static equilibrium conditions.

To this end, as a conceptual motivation, we note on the one hand that physical forces in the sense of Newton together with their first order moments are generated by variations relative to the ambient physical (euclidian) space $\mathbb{E}^3$ at fixed position in material space. Accordingly, on the other hand, material forces in the sense of Eshelby together with their first order moments are generated by variations relative to the ambient material (manifold) space $\mathbb{M}^3$ at fixed position in physical space. In the following quasi-static equilibrium conditions for physical and material forces together with the corresponding conditions for their first order moments are stated in global form. Thereby we assume that physical and material surface tractions and volume forces are given, ignoring the fact that the material forces are only known a posteriori, i.e. when the direct motion problem has already been solved. Moreover, the detailed outline of the format of the Eshelby stress and the material volume force is postponed until the next section.

3.1. Physical viewpoint of direct motion description

First, we consider an arbitrary subdomain $\mathscr{V}$ with boundary $\partial\mathscr{V}$ of the current configuration $\mathscr{B}$ (Fig. 3). We assume that the subdomain is loaded along $\partial\mathscr{V}$ by physical surface tractions in terms of the spatial Cauchy stress $\boldsymbol{\sigma}$ and the current surface normal $\boldsymbol{n}$ and within $\mathscr{V}$ by physical volume forces in terms of distributed physical volume forces $\boldsymbol{b}^{\text{phy}}$, e.g. gravity.

3.1.1. Physical forces

Therefore, we may define the resultant physical surface and volume forces acting on $\mathscr{V}$ as

$$\boldsymbol{F}^{\text{phy,sur}} = \int_{\partial\mathscr{V}} \boldsymbol{\sigma}^t \cdot \boldsymbol{n}\,\mathrm{d}a \quad \text{and} \quad \boldsymbol{F}^{\text{phy,vol}} = \int_{\mathscr{V}} \boldsymbol{b}^{\text{phy}}\,\mathrm{d}v. \tag{1}$$

Then, the familiar statement of quasi-static equilibrium of physical forces for the subdomain with current configuration $\mathscr{V}$ writes along the lines of Newton and Cauchy as

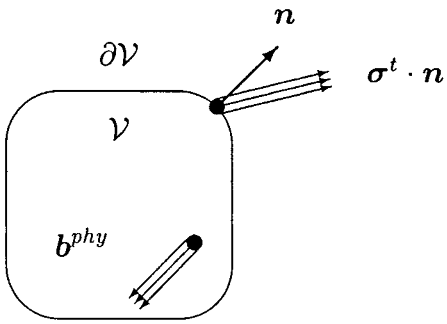

Fig. 3. Physical viewpoint of direct motion description: physical surface tractions acting on the boundary and physical volume forces acting in the bulk.

$$\boldsymbol{F}^{\text{phy,sur}} + \boldsymbol{F}^{\text{phy,vol}} = \mathbf{0}. \tag{2}$$

3.1.2. Vectorial moment of physical force

Likewise, with $\boldsymbol{r} \in \mathbb{E}^3$ denoting the distance vector to a fixed point in (3d) physical space and $\text{axl}[\bullet] = -\frac{1}{2}[\bullet] \times \boldsymbol{I}$ denoting the axial vector of a second order tensor $[\bullet]$ in $\mathscr{V}$ we may define the resultant vectorial moment of physical surface and volume forces acting on $\mathscr{V}$ as

$$\boldsymbol{T}^{\text{phy,sur}} = \int_{\partial\mathscr{V}} \boldsymbol{r} \times \boldsymbol{\sigma}^t \cdot \boldsymbol{n}\,\mathrm{d}a, \tag{3a}$$

$$\boldsymbol{T}^{\text{phy,vol}} = \int_{\mathscr{V}} \left[\boldsymbol{r} \times \boldsymbol{b}^{\text{phy}} + 2\text{axl}\,\boldsymbol{\sigma}\right] \mathrm{d}v. \tag{3b}$$

Vectorial moments of physical forces are commonly denoted as (physical) torques and are essentially a measure of the noncentrality of the physical force system acting on $\mathscr{V}$. Thus the statement equivalent to the quasi-static equilibrium of the noncentral part of physical forces for the subdomain with current configuration $\mathscr{V}$ writes

$$\boldsymbol{T}^{\text{phy,sur}} + \boldsymbol{T}^{\text{phy,vol}} = \mathbf{0}. \tag{4}$$

Please recall that the condition to obtain an independent equilibrium equation along the lines of Euler is the symmetry of $\boldsymbol{\sigma}$, since only then the volume contribution $2\text{axl}\,\boldsymbol{\sigma}$ in Eq. (3b) vanishes. The discussion of the condition for $\boldsymbol{\sigma}$ being symmetric is postponed until the next section.

3.1.3. Scalar moment of physical force

Moreover, with $\text{prs}[\bullet] = -\frac{1}{3}[\bullet] : \boldsymbol{I}$ denoting the pressure part of a second order tensor $[\bullet]$ in $\mathscr{V}$ we may define the resultant scalar moment of physical surface and volume forces acting on $\mathscr{V}$ as

$$V^{\text{phy,sur}} = \int_{\partial\mathscr{V}} \boldsymbol{r} \cdot \boldsymbol{\sigma}^t \cdot \boldsymbol{n}\,\mathrm{d}a \quad \text{and} \quad V^{\text{phy,vol}} = \int_{\mathscr{V}} \left[\boldsymbol{r} \cdot \boldsymbol{b}^{\text{phy}} + 3\text{prs}\,\boldsymbol{\sigma}\right] \mathrm{d}v. \tag{5}$$

Scalar moments of physical forces along the lines of Möbius are sometimes denoted as (physical) virials and are essentially a measure of the centrality of the physical force system acting on $\mathscr{V}$. Thus, the statement equivalent to the quasi-static equilibrium of the central part of physical forces for the subdomain with current configuration $\mathscr{V}$ writes as

$$V^{\text{phy,sur}} + V^{\text{phy,vol}} = 0. \tag{6}$$

Please note that Eq. (6) takes the character of an independent equilibrium equation if $\boldsymbol{\sigma}$ is always purely deviatoric since only then the contribution $3\mathrm{prs}\,\boldsymbol{\sigma}$ vanishes.

3.1.4. Dyadic moment of physical force

Finally, with the somewhat redundant abbreviation $\mathrm{neg}[\bullet] = -[\bullet]$ for the negative part of a second order tensor $[\bullet]$ in $\mathscr{V}$, we may define the resultant dyadic moment of physical surface and volume forces acting on $\mathscr{V}$ as

$$\boldsymbol{D}^{\mathrm{phy,sur}} = \int_{\partial\mathscr{V}} \boldsymbol{r} \otimes \boldsymbol{\sigma}^t \cdot \boldsymbol{n}\,\mathrm{d}a \quad \text{and} \quad \boldsymbol{D}^{\mathrm{phy,vol}} = \int_{\mathscr{V}} \left[\boldsymbol{r} \otimes \boldsymbol{b}^{\mathrm{phy}} + \mathrm{neg}\,\boldsymbol{\sigma}\right] \mathrm{d}v. \tag{7}$$

Observe however that the skew-symmetric contributions to Eq. (7) correspond to the contributions to Eqs. (3a,b). The statement equivalent to the quasi-static equilibrium of physical forces for the subdomain with current configuration $\mathscr{V}$ writes as

$$\boldsymbol{D}^{\mathrm{phy,sur}} + \boldsymbol{D}^{\mathrm{phy,vol}} = \boldsymbol{0}. \tag{8}$$

For vanishing physical volume forces $\boldsymbol{b}^{\mathrm{phy}}$, Eq. (8) simply relates the average of $\boldsymbol{\sigma}$ in $\mathscr{V}$ to the physical surface forces acting on $\partial\mathscr{V}$.

3.2. Material viewpoint of inverse motion description

Next we consider an arbitrary subdomain $\mathscr{V}_0$ with boundary $\partial\mathscr{V}_0$ of the reference configuration $\mathscr{B}_0$ (Fig. 4). We assume that the subdomain is loaded along $\partial\mathscr{V}_0$ by material surface tractions in terms of the material Eshelby stress $\boldsymbol{M}$ and the reference surface normal $\boldsymbol{N}$ and within $\mathscr{V}_0$ by material volume forces in terms of distributed material volume forces $\boldsymbol{B}^{\mathrm{mat}}$, stemming e.g. from material inhomogeneities.

Please observe that the material surface tractions and material volume forces are only obtained as part of the solution, since they contain information about the deformation as will be lined out in Section 3.2.1.

3.2.1. Material forces

As before, we may define the resultant material surface and volume forces acting on $\mathscr{V}_0$ as

$$\boldsymbol{F}^{\mathrm{mat,sur}} = \int_{\partial\mathscr{V}_0} \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A \quad \text{and} \quad \boldsymbol{F}^{\mathrm{mat,vol}} = \int_{\mathscr{V}_0} \boldsymbol{B}^{\mathrm{mat}}\,\mathrm{d}V. \tag{9}$$

Then, the somewhat unusual statement of quasi-static equilibrium of material forces for the subdomain with reference configuration $\mathscr{V}_0$ writes along the lines of Eshelby as

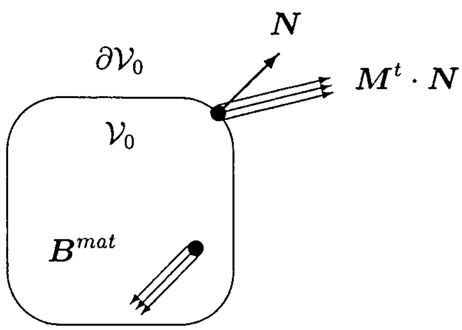

Fig. 4. Material viewpoint of inverse motion description: material surface tractions acting on the boundary and material volume forces acting in the bulk.

$$\boldsymbol{F}^{\mathrm{mat,sur}} + \boldsymbol{F}^{\mathrm{mat,vol}} = \boldsymbol{0}. \tag{10}$$

3.2.2. Vectorial moment of material forces

Likewise, with $\boldsymbol{R} \in \mathbb{M}^3$ denoting the distance vector to a fixed point in material space and $\mathrm{Axl}[\bullet] = -\frac{1}{2}[\bullet] \times \boldsymbol{I}$ denoting the axial vector of a second order tensor $[\bullet]$ in $\mathscr{V}_0$ we may define the resultant vectorial moment of material surface and volume forces acting on $\mathscr{V}_0$ as

$$\boldsymbol{T}^{\mathrm{mat,sur}} = \int_{\partial\mathscr{V}_0} \boldsymbol{R} \times \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A, \tag{11a}$$

$$\boldsymbol{T}^{\mathrm{mat,vol}} = \int_{\mathscr{V}_0} [\boldsymbol{R} \times \boldsymbol{B}^{\mathrm{mat}} + 2\mathrm{Axl}\,\boldsymbol{M}]\,\mathrm{d}V. \tag{11b}$$

Note that our definition of the vectorial moment of material forces based on the material vector $\boldsymbol{R}$ is in contrast to the definition advocated by Maugin (1993) which is based on the vector $\boldsymbol{C} \cdot \boldsymbol{R}$. Vectorial moments of material forces may be denoted as material torques and are essentially a measure of the noncentrality of the material force system acting on $\mathscr{V}_0$.

Thus, the statement equivalent to the quasi-static equilibrium of the noncentral part of the material forces for the subdomain with reference configuration $\mathscr{V}_0$ is written as

$$\boldsymbol{T}^{\mathrm{mat,sur}} + \boldsymbol{T}^{\mathrm{mat,vol}} = \boldsymbol{0}. \tag{12}$$

Please note that the volume contribution $2\mathrm{Axl}\,\boldsymbol{M}$ to Eq. (11b) vanishes only for symmetric $\boldsymbol{M}$. The discussion of the condition for $\boldsymbol{M}$ being symmetric is postponed until the next section.

3.2.3. Scalar moment of material forces

Moreover, with $\mathrm{prs}[\bullet] = -\frac{1}{3}[\bullet] : \boldsymbol{I}$ denoting the pressure part of a second order tensor $[\bullet]$ in $\mathscr{V}_0$ we may define the resultant scalar moment of material surface and volume forces acting on $\mathscr{V}_0$ as

$$V^{\mathrm{mat,sur}} = \int_{\partial\mathscr{V}_0} \boldsymbol{R} \cdot \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A \quad \text{and} \quad V^{\mathrm{mat,vol}} = \int_{\mathscr{V}_0} [\boldsymbol{R} \cdot \boldsymbol{B}^{\mathrm{mat}} + 3\mathrm{prs}\,\boldsymbol{M}]\,\mathrm{d}V. \tag{13}$$

In analogy to Eq. (5), the scalar moment of material forces may be denoted as material virial and serves as a measure of the centrality of the material force system acting on $\mathscr{V}_0$. Thus, the statement equivalent to the quasi-static equilibrium of the central part of material forces for the subdomain with reference configuration $\mathscr{V}_0$ writes as

$$V^{\mathrm{mat,sur}} + V^{\mathrm{mat,vol}} = 0. \tag{14}$$

Please note that the contribution $3\mathrm{prs}\,\boldsymbol{M}$ to the physical volume virial vanishes only for deviatoric $\boldsymbol{M}$, in this case making Eq. (14) an independent equilibrium equation.

3.2.4. Dyadic moment of material forces

Finally, with the abbreviation $\mathrm{Neg}\,[\bullet] = -[\bullet]$ for the negative part of a second order tensor $[\bullet]$ in $\mathscr{V}_0$, we may define the resultant dyadic moment of material surface and volume forces as

$$\boldsymbol{D}^{\mathrm{mat,sur}} = \int_{\partial\mathscr{V}_0} \boldsymbol{R} \otimes \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A \quad \text{and} \quad \boldsymbol{D}^{\mathrm{mat,vol}} = \int_{\mathscr{V}_0} [\boldsymbol{R} \otimes \boldsymbol{B}^{\mathrm{mat}} + \mathrm{Neg}\;\boldsymbol{M}]\,\mathrm{d}V. \tag{15}$$

Observe that the skew-symmetric contributions to Eq. (15) correspond to the contributions to Eq. (11a,b). Then, the statement equivalent to the quasi-static equilibrium of material forces for the subdomain with reference configuration $\mathscr{V}_0$ writes as

$$\boldsymbol{D}^{\mathrm{mat,sur}} + \boldsymbol{D}^{\mathrm{mat,vol}} = \boldsymbol{0}. \tag{16}$$

Accordingly, for vanishing material volume forces $\boldsymbol{B}^{\mathrm{mat}}$, this equation relates the average of $\boldsymbol{M}$ in $\mathscr{V}_0$ to the material surface forces acting on $\partial\mathscr{V}_0$.

4. Quasi-static balance equations

Before discussing the appropriate quasi-static balance equations for a conservative system with hyperelastic characterization of the material response it remains to specify the physical and material stress measures $\boldsymbol{\sigma}$ and $\boldsymbol{M}$, which we shall denote the spatial Newton stress and the material Eshelby stress, respectively, together with the physical and material distributed body forces $\boldsymbol{b}^{\mathrm{phy}}$ and $\boldsymbol{B}^{\mathrm{mat}}$.

4.1. Hyperelastic characterization of material behavior

For a hyperelastic material response the stored energy density per unit volume in $\mathscr{B}_0$ and per unit volume in $\mathscr{B}$ is given by $\mathscr{W}_0 = \mathscr{W}_0(\boldsymbol{F};\boldsymbol{X})$ and $\mathscr{W} = \mathscr{W}(\boldsymbol{f};\boldsymbol{X})$, respectively. Here $([\bullet];\boldsymbol{X})$ denotes an explicit dependence on the material position $\boldsymbol{X}$, as an example consider an inhomogeneous distribution of material constants over $\mathscr{B}_0$. An alternative explicit dependence on the spatial position $\boldsymbol{x}$ does not make sense from the lagrangian viewpoint where either the direct or the inverse motion of the material particles is observed, i.e. the material particle is considered as being the carrier of physical properties.

On the one hand, physical objectivity, i.e. the common notion of frame indifference, requires invariant response in stored energy density upon superposition of a rigid body motion onto $\mathscr{B}$ with $\boldsymbol{F} \to \boldsymbol{F}^* = \boldsymbol{Q}\cdot\boldsymbol{F}$ or $\boldsymbol{f} \to \boldsymbol{f}^* = \boldsymbol{f}\cdot\boldsymbol{Q}^t$, respectively, whereby $\boldsymbol{Q} \in \mathrm{SO}(3)$ denotes an arbitrary rotation. Thus, physical objectivity restricts the dependence of $\mathscr{W}_0$ on $\boldsymbol{F}$ to $\mathscr{W}_0 = \mathscr{W}_0(\boldsymbol{C};\boldsymbol{X})$ and the dependence of $\mathscr{W}$ on $\boldsymbol{f}$ to $\mathscr{W} = \mathscr{W}(\boldsymbol{B};\boldsymbol{X})$, respectively.

On the other hand material isotropy, with which we shall denote material objectivity in the sequel, requires invariant response in stored energy density upon superposition of a rigid body motion onto $\mathscr{B}_0$ with $\boldsymbol{f} \to \boldsymbol{f}^* = \boldsymbol{q}\cdot\boldsymbol{f}$ or $\boldsymbol{F} \to \boldsymbol{F}^* = \boldsymbol{F}\cdot\boldsymbol{q}^t$, respectively, whereby $\boldsymbol{q} \in \mathrm{SO}(3)$ denotes an arbitrary rotation. Thus, material objectivity restricts the dependence of $\mathscr{W}_0$ on $\boldsymbol{F}$ to $\mathscr{W}_0 = \mathscr{W}_0(\boldsymbol{b};\boldsymbol{X})$ and the dependence of $\mathscr{W}$ on $\boldsymbol{f}$ to $\mathscr{W} = \mathscr{W}(\boldsymbol{c};\boldsymbol{X})$, respectively.

It should be noted that physical objectivity is a general requirement and is therefore mandatory, whereas material objectivity relates to a specific material response and is therefore optional. The consequences of the different qualities of these requirements on the symmetry of the physical and material stress measures are highlighted below.

4.1.1. Hyperelastic stress measures of Newton type

In the direct motion setting, the two-point Newton stress and the spatial Newton stress in Fig. 5, which are called the (direct motion) 1. Piola–Kirchhoff stress and the (direct motion) Cauchy stress in common terminology, follow as

$$\boldsymbol{\Sigma}^t = \partial_{\boldsymbol{F}}\mathscr{W}_0, \tag{17a}$$

$$\boldsymbol{\sigma}^t = j\boldsymbol{\Sigma}^t\cdot\boldsymbol{F}^t = \mathscr{W}\boldsymbol{I} - \boldsymbol{f}^t\cdot\partial_{\boldsymbol{f}}\mathscr{W}. \tag{17b}$$

Thereby, the last expression in Eq. (17a,b) denotes the energy-momentum format of the spatial Newton stress, whereby, without danger of confusion, we omitted the explicit indication of the direct or inverse motion parametrization for the sake of conciseness.

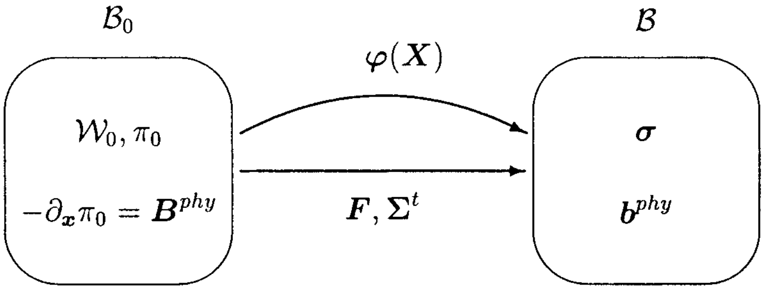

Fig. 5. Direct motion setting – Newton type 1. Piola–Kirchhoff and Cauchy stress derived from stored energy density together with physical volume forces derived from bulk potential energy density.

Please note that on the one hand, symmetry of the spatial Newton stress $\boldsymbol{\sigma}$ with respect to the standard euclidian metric is mandatory since it is a consequence of the underlying requirement of physical objectivity or rather frame indifference with $\mathscr{W}_0 = \mathscr{W}_0(\boldsymbol{C}; \boldsymbol{X})$ and $\mathscr{W} = \mathscr{W}(\boldsymbol{B}; \boldsymbol{X})$, thus, we encounter symmetric expressions for

$$\partial_{\boldsymbol{F}} \mathscr{W}_0 \cdot \boldsymbol{F}^t = 2\boldsymbol{F} \cdot \partial_{\boldsymbol{C}} \mathscr{W}_0 \cdot \boldsymbol{F}^t \quad \text{and} \quad \boldsymbol{f}^t \cdot \partial_{\boldsymbol{f}} \mathscr{W} = 2\boldsymbol{f}^t \cdot \partial_{\boldsymbol{B}} \mathscr{W} \cdot \boldsymbol{f} \tag{18}$$

which enter the definition of the spatial Newton stress $\boldsymbol{\sigma}$ in Eqs. (17a) and (17b). On the other hand symmetry of the spatial Newton stress $\boldsymbol{\sigma}$ with respect to the strain metric $\boldsymbol{c}$, i.e. $\boldsymbol{\sigma}^t \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{\sigma}$, is only optional as a consequence of the requirement of material objectivity or rather material isotropy, for a proof refer to Eq. (20).

4.1.2. Hyperelastic stress measures of Eshelby type

In the inverse motion setting, the two-point Eshelby stress and the material Eshelby stress in Fig. 6, which might be called the inverse motion 1. Piola–Kirchhoff stress and the inverse motion Cauchy stress, follow as

$$\boldsymbol{m}^t = \partial_{\boldsymbol{f}} \mathscr{W} \tag{19a}$$

$$\boldsymbol{M}^t = J\boldsymbol{m}^t \cdot \boldsymbol{f}^t = \mathscr{W}_0 \boldsymbol{I} - \boldsymbol{F}^t \cdot \partial_{\boldsymbol{F}} \mathscr{W}_0. \tag{19b}$$

Thereby, the last expression in Eq. (19a) and (19b) denotes the energy-momentum format of the material Eshelby stress, whereby, without danger of confusion, we again omitted the explicit indication of the direct or inverse motion parametrization for the sake of conciseness.

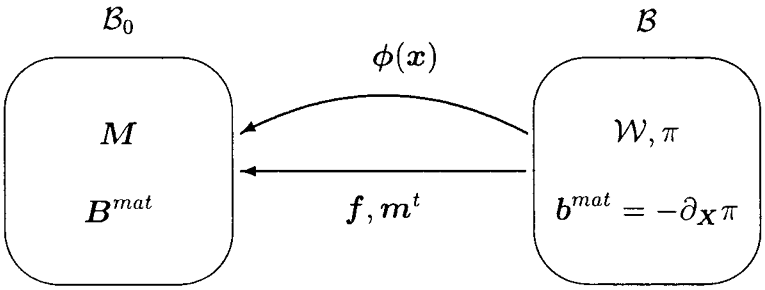

Fig. 6. Inverse motion setting – Eshelby type 1. Piola–Kirchhoff and Cauchy stress derived from stored energy density together with material volume forces derived from bulk potential energy density.

Please note on the one hand that symmetry of the material Eshelby stress $\boldsymbol{M}$ with respect to the standard euclidian metric is only optional as a consequence of the requirement of material objectivity or rather material isotropy with $\mathscr{W}_0 = \mathscr{W}_0(\boldsymbol{b}; \boldsymbol{X})$ and $\mathscr{W} = \mathscr{W}(\boldsymbol{c}; \boldsymbol{X})$; thus, only in this particular case, we encounter symmetric expressions for

$$\partial_{\boldsymbol{f}}\mathscr{W} \cdot \boldsymbol{f}^t = 2\boldsymbol{f} \cdot \partial_{\boldsymbol{c}}\mathscr{W} \cdot \boldsymbol{f}^t \quad \text{and} \quad \boldsymbol{F}^t \cdot \partial_{\boldsymbol{F}}\mathscr{W}_0 = 2\boldsymbol{F}^t \cdot \partial_{\boldsymbol{b}}\mathscr{W}_0 \cdot \boldsymbol{F} \tag{20}$$

which enter the definition of the material Eshelby stress $\boldsymbol{M}$ in Eq. (19a) and (19b). On the other hand, symmetry of the material Eshelby stress $\boldsymbol{M}$ with respect to the strain metric $\boldsymbol{C}$, i.e. $\boldsymbol{M}^t \cdot \boldsymbol{C} = \boldsymbol{C} \cdot \boldsymbol{M}$, is mandatory since it is a consequence of the requirement of physical objectivity or rather frame indifference, for a proof refer to Eq. (18).

4.2. Physical quasi-static balance equations

4.2.1. Balance of momentum

Firstly, localizing Eq. (2) renders the familiar quasi-static balance of physical momentum commonly attributed to Cauchy as

$$-\mathrm{Div}\,\boldsymbol{\Sigma}^t = \boldsymbol{B}^{\mathrm{phy}} \iff -\mathrm{div}\,\boldsymbol{\sigma}^t = \boldsymbol{b}^{\mathrm{phy}}. \tag{21}$$

Here, distributed physical volume forces $\boldsymbol{B}^{\mathrm{phy}} = -\partial_{\boldsymbol{x}}\pi_0$ follow from the explicit physical variation of the bulk potential energy density $\pi_0 = \mathscr{W}_0([\bullet]; \boldsymbol{X}) - \boldsymbol{\varphi} \cdot \boldsymbol{B}^{\mathrm{phy}}$ per unit volume in $\mathscr{B}_0$. Please observe that the physical volume force $\boldsymbol{B}^{\mathrm{phy}}$ per unit volume in $\mathscr{B}_0$ is considered to be given for the direct motion description and therefore does not change with a physical variation at fixed $\boldsymbol{X}$. Moreover, $\partial_{\boldsymbol{x}}\mathscr{W}_0 = \boldsymbol{0}$ holds for the explicit physical gradient of the stored energy density. It shall be noted carefully that the vectorial residuum of the quasi-static balance of physical momentum has components in physical space. Moreover, please recall that the quasi-static balance of physical momentum serves for solving the direct motion problem.

4.2.2. Balance of vectorial moment of momentum

Then, localizing Eq. (4) renders additionally the quasi-static balance of vectorial moment of physical momentum as

$$-\mathrm{Div}(\boldsymbol{r} \times \boldsymbol{\Sigma}^t) = \boldsymbol{r} \times \boldsymbol{B}^{\mathrm{phy}} + 2\mathrm{axl}\ (J\boldsymbol{\sigma}) \iff -\mathrm{div}(\boldsymbol{r} \times \boldsymbol{\sigma}^t) = \boldsymbol{r} \times \boldsymbol{b}^{\mathrm{phy}} + 2\mathrm{axl}\ \boldsymbol{\sigma}. \tag{22}$$

Taking into account the mandatory symmetry of the spatial Newton stress $\boldsymbol{\sigma}$ with respect to the standard euclidian metric due to physical objectivity, Eq. (4) and thus Eq. (22) reduces to the well-known quasi-static format of equilibrium of physical torques.

4.2.3. Balance of scalar moment of momentum

Likewise, localizing Eq. (6) renders additionally the quasi-static 'balance' of scalar moment of physical momentum as

$$-\mathrm{Div}(\boldsymbol{r} \cdot \boldsymbol{\Sigma}^t) = \boldsymbol{r} \cdot \boldsymbol{B}^{\mathrm{phy}} + 3\mathrm{prs}\ (J\boldsymbol{\sigma}) \iff -\mathrm{div}(\boldsymbol{r} \cdot \boldsymbol{\sigma}^t) = \boldsymbol{r} \cdot \boldsymbol{b}^{\mathrm{phy}} + 3\mathrm{prs}\ \boldsymbol{\sigma}. \tag{23}$$

For vanishing physical volume forces $\boldsymbol{b}^{\mathrm{phy}} = \boldsymbol{0}$, Eq. (6) reduces to a simple averaging equation for the pressure part of the spatial Newton stress $\boldsymbol{\sigma}$. Recalling the dynamical format of the virial theorem of classical mechanics, which relates basically the long time average of the kinetic and the potential energy for conservative systems, we conclude that in the continuum setting, Eq. (23) may be denoted as the quasi-static format of the virial theorem of physical momentum.

4.2.4. Balance of dyadic moment of momentum

Finally, localizing Eq. (8) renders additionally the quasi-static 'balance' of dyadic moment of physical momentum as

$$-\mathrm{Div}(\boldsymbol{r}\otimes\boldsymbol{\Sigma}^t)=\boldsymbol{r}\otimes\boldsymbol{B}^{\mathrm{phy}}+\mathrm{neg}\ (J\boldsymbol{\sigma})\iff -\mathrm{div}(\boldsymbol{r}\otimes\boldsymbol{\sigma}^t)=\boldsymbol{r}\otimes\boldsymbol{b}^{\mathrm{phy}}+\mathrm{neg}\ \boldsymbol{\sigma}. \tag{24}$$

For vanishing physical volume forces $\boldsymbol{b}^{\mathrm{phy}}=\boldsymbol{0}$, Eq. (8) and thus Eq. (24) reduces to a simple averaging equation for the spatial Newton stress $\boldsymbol{\sigma}$.

4.3. Material quasi-static balance equations

4.3.1. Balance of momentum

Next, localizing Eq. (10) renders the quasi-static balance of pseudomomentum commonly attributed to Eshelby, see the monograph by Maugin (1993) for a detailed justification of the terminology pseudomomentum, as

$$-\mathrm{Div}\,\boldsymbol{M}^t=\boldsymbol{B}^{\mathrm{mat}}\iff -\mathrm{div}\,\boldsymbol{m}^t=\boldsymbol{b}^{\mathrm{mat}}. \tag{25}$$

Here, distributed material volume forces $\boldsymbol{b}^{\mathrm{mat}}=-\partial_{\boldsymbol{X}}\pi$ with $\boldsymbol{b}^{\mathrm{mat}}=-\partial_{\boldsymbol{X}}\mathscr{W}-\boldsymbol{F}^t\cdot\boldsymbol{b}^{\mathrm{phy}}$ follow from the explicit material variation of the bulk potential energy density $\pi=\mathscr{W}([\bullet];\boldsymbol{\phi})-\boldsymbol{\phi}^{-1}\cdot\boldsymbol{b}^{\mathrm{phy}}$ per unit volume in $\mathscr{B}$. Please observe that the physical volume force $\boldsymbol{b}^{\mathrm{phy}}$ per unit volume in $\mathscr{B}$ is considered to be given for the inverse motion description and therefore does not change with a material variation at fixed $\boldsymbol{x}$. Moreover, since $\mathscr{W}=j\mathscr{W}_0$, the relation $\partial_{\boldsymbol{X}}\mathscr{W}=j\partial_{\boldsymbol{X}}\mathscr{W}_0$ holds for the explicit material gradient of the stored energy density.

It shall be noted carefully that the vectorial residuum of the quasi-static balance of pseudomomentum has components in material space. Moreover, please recall that the quasi-static balance of pseudomomentum serves for solving the inverse motion problem.

4.3.2. Balance of vectorial moment of momentum

Then, localizing Eq. (12) renders additionally the quasi-static balance of vectorial moment of pseudomomentum as

$$-\mathrm{Div}(\boldsymbol{R}\times\boldsymbol{M}^t)=\boldsymbol{R}\times\boldsymbol{B}^{\mathrm{mat}}+2\mathrm{Axl}\ \boldsymbol{M}\iff -\mathrm{div}(\boldsymbol{R}\times\boldsymbol{m}^t)=\boldsymbol{R}\times\boldsymbol{b}^{\mathrm{mat}}+2\mathrm{Axl}\ (j\boldsymbol{M}). \tag{26}$$

Taking into account the optional symmetry of the material Eshelby stress $\boldsymbol{M}$ with respect to the standard euclidian metric due to material objectivity, in this case Eq. (12) and thus Eq. (26) reduce to the intuitive quasi-static format of equilibrium of material torques.

4.3.3. Balance of scalar moment of momentum

Likewise, localizing Eq. (14) renders additionally the quasi-static 'balance' of scalar moment of pseudomomentum as

$$-\mathrm{Div}(\boldsymbol{R}\cdot\boldsymbol{M}^t)=\boldsymbol{R}\cdot\boldsymbol{B}^{\mathrm{mat}}+3\mathrm{prs}\ \boldsymbol{M}\iff -\mathrm{div}(\boldsymbol{R}\cdot\boldsymbol{m}^t)=\boldsymbol{R}\cdot\boldsymbol{b}^{\mathrm{mat}}+3\mathrm{prs}\ (j\boldsymbol{M}). \tag{27}$$

For vanishing physical volume forces $\boldsymbol{B}^{\mathrm{mat}}=\boldsymbol{0}$, Eq. (14) reduces to a simple averaging equation for the pressure of the material Eshelby stress $\boldsymbol{M}$.

4.3.4. Balance of dyadic moment of momentum

Finally, localizing Eq. (16) renders additionally the quasi-static 'balance' of dyadic moment of pseudomomentum as

$$-\mathrm{Div}(\boldsymbol{R}\otimes\boldsymbol{M}^t)=\boldsymbol{R}\otimes\boldsymbol{B}^{\mathrm{mat}}+\mathrm{Neg}\ \boldsymbol{M}\iff -\mathrm{div}(\boldsymbol{R}\otimes\boldsymbol{m}^t)=\boldsymbol{R}\otimes\boldsymbol{b}^{\mathrm{mat}}+\mathrm{Neg}\ (j\boldsymbol{M}). \tag{28}$$

For vanishing physical volume forces $\boldsymbol{B}^{\text{mat}} = \boldsymbol{0}$, Eq. (16) and thus Eq. (28) reduces to a simple averaging equation for the material Eshelby stress $\boldsymbol{M}$.

Remark 4.1. *For a proof of the last expression in Eq. (17a) and (17b), which denotes the energy-momentum format of the spatial Newton stress, compute the push forward of*

$$\boldsymbol{\Sigma}^t = \partial_{\boldsymbol{F}}[J\mathscr{W}] = J\mathscr{W}\boldsymbol{F}^{-t} + J\partial_{\boldsymbol{F}}\mathscr{W} = J\mathscr{W}\boldsymbol{F}^{-t} + J\partial_{\boldsymbol{f}}\mathscr{W} : \partial_{\boldsymbol{F}}\boldsymbol{f} = J\mathscr{W}\boldsymbol{F}^{-t} - J\boldsymbol{F}^{-t}\cdot\partial_{\boldsymbol{f}}\mathscr{W}\cdot\boldsymbol{F}^{-t}.$$

In analogy, for a proof of the last expression in Eq. (19a) and (19b), which denotes the energy-momentum format of the material Eshelby stress, compute the pull back of

$$\boldsymbol{m}^t = \partial_{\boldsymbol{f}}[j\mathscr{W}_0] = j\mathscr{W}_0\boldsymbol{f}^{-t} + j\partial_{\boldsymbol{f}}\mathscr{W}_0 = j\mathscr{W}_0\boldsymbol{f}^{-t} + j\partial_{\boldsymbol{F}}\mathscr{W}_0 : \partial_{\boldsymbol{f}}\boldsymbol{F} = j\mathscr{W}_0\boldsymbol{f}^{-t} - j\boldsymbol{f}^{-t}\cdot\partial_{\boldsymbol{F}}\mathscr{W}_0\cdot\boldsymbol{f}^{-t}.$$

Remark 4.2. *The relation between the quasi-static balances of physical and pseudomomentum may be highlighted by observing the following identities:*

$$-\boldsymbol{F}^t\cdot\operatorname{Div}(\partial_{\boldsymbol{F}}\mathscr{W}_0) = \operatorname{Div}\boldsymbol{M}^t - \partial_{\boldsymbol{X}}\mathscr{W}_0 \quad \text{and} \quad -\boldsymbol{f}^t\cdot\operatorname{div}(\partial_{\boldsymbol{f}}\mathscr{W}) = \operatorname{div}\boldsymbol{\sigma}^t - \boldsymbol{f}^t\cdot\partial_{\boldsymbol{X}}\mathscr{W}.$$

For a proof of these identities, we assume sufficiently smooth motions and incorporate the integrability conditions for $\boldsymbol{F}$ and $\boldsymbol{f}$ as given in Section 2.

Remark 4.3. *The quasi-static 'balances' of scalar moments of physical and pseudomomentum may be related by taking into account the following identities*:

$$\mathscr{W}_0 = -[\operatorname{prs}(J\boldsymbol{\sigma}) + \operatorname{prs}\boldsymbol{M}] \quad \text{and} \quad \mathscr{W} = -[\operatorname{prs}\boldsymbol{\sigma} + \operatorname{prs}(j\boldsymbol{M})]$$

which follow directly from the energy-momentum formats of the spatial Newton and the material Eshelby stress in Eqs. (17b) and (19b), respectively.

Remark 4.4. *As a final remark we note that we may introduce two alternative stress measures by taking into account the displacement field $\boldsymbol{u} = \boldsymbol{\varphi} - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{\phi}$ and thus by defining the displacement gradients $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}$ and $\boldsymbol{h} = \boldsymbol{I} - \boldsymbol{f}$. With these relations at hand the material Newton–Eshelby stress $\tilde{\boldsymbol{M}}$, which is in fact most often used in geometrically linear fracture mechanics instead of $\boldsymbol{M}$, and the spatial Newton–Eshelby stress $\tilde{\boldsymbol{m}}$ might be introduced as*

$$\tilde{\boldsymbol{M}}^t = \boldsymbol{M}^t + \boldsymbol{\Sigma}^t = \mathscr{W}_0\boldsymbol{I} - \boldsymbol{H}^t\cdot\partial_{\boldsymbol{F}}\mathscr{W}_0 \quad \text{and} \quad \tilde{\boldsymbol{m}}^t = \boldsymbol{\sigma}^t + \boldsymbol{m}^t = \mathscr{W}\boldsymbol{I} + \boldsymbol{h}^t\cdot\partial_{\boldsymbol{f}}\mathscr{W}.$$

Consequently, a corresponding balance of momentum is stated as

$$-\operatorname{Div}\tilde{\boldsymbol{M}}^t = \tilde{\boldsymbol{B}} \iff -\operatorname{div}\tilde{\boldsymbol{m}}^t = \tilde{\boldsymbol{b}}.$$

Here, the distributed volume forces follow as $\tilde{\boldsymbol{B}} = \boldsymbol{B}^{\text{mat}} + \boldsymbol{B}^{\text{phy}}$ and $\tilde{\boldsymbol{b}} = \boldsymbol{b}^{\text{phy}} + \boldsymbol{b}^{\text{mat}}$, respectively. Indeed, as commented on by Chadwick (1975), *Eshelby essentially employed $\tilde{\boldsymbol{M}}$ rather than $\boldsymbol{M}$ in* Eshelby (1951, 1956, 1970, 1975). *Nevertheless, $\tilde{\boldsymbol{M}}$ has the obvious drawback that it is generally non-symmetric with respect to the standard euclidian metric.*

5. Variational form of quasi-static balance equations

It is interesting to recast the quasi-static balances of physical and pseudomomentum in their weak or rather variational form in order to study the energetic contributions of the different quantities involved. Moreover, the duality of the direct and the inverse motion point of view is highlighted once again. Finally, weak forms constitute the natural point of departure for discretization methods as described in Part II of this work.

5.1. Physical quasi-static balance equations

Firstly, the pointwise statement in Eq. (21) for the solution of the direct problem is tested by a test function (physical virtual displacement) $\boldsymbol{w}$ under the necessary smoothness and boundary assumptions to render the well-known virtual work expression

$$\underbrace{\int_{\partial\mathscr{B}} \boldsymbol{w}\cdot\boldsymbol{\sigma}^t\cdot\boldsymbol{n}\,\mathrm{d}a}_{S^{\mathrm{sur}}} = \underbrace{\int_{\mathscr{B}} \nabla_x\boldsymbol{w} : \boldsymbol{\sigma}^t\,\mathrm{d}v}_{S^{\mathrm{int}}} - \underbrace{\int_{\mathscr{B}} \boldsymbol{w}\cdot\boldsymbol{b}^{\mathrm{phy}}\,\mathrm{d}v}_{S^{\mathrm{vol}}} \quad \forall\,\boldsymbol{w}. \tag{29}$$

The implications of this variational statement, in particular the energetic interpretation of S^{sur}, S^{int} and S^{vol}, are discussed in the sequel.

5.1.1. Energetic interpretation

The different energetic terms in Eq. (29) may be interpreted by considering the physical variation at fixed $\boldsymbol{X}$ of the total bulk potential energy, i.e. the quasi-static action integral

$$\begin{aligned}
\delta_X \int_{\mathscr{B}_0} \pi_0\,\mathrm{d}V &= \int_{\mathscr{B}_0} \left[\delta_X \mathscr{W}_0 - \boldsymbol{B}^{\mathrm{phy}}\cdot\delta\boldsymbol{\varphi}\right]\mathrm{d}V,\\
&= \int_{\mathscr{B}_0} \left[\partial_{\boldsymbol{F}} \mathscr{W}_0 : \nabla_X\delta\boldsymbol{\varphi} - \boldsymbol{B}^{\mathrm{phy}}\cdot\delta\boldsymbol{\varphi}\right]\mathrm{d}V,\\
&= \int_{\mathscr{B}_0} \left[\boldsymbol{\Sigma}^t : \nabla_X\delta\boldsymbol{\varphi} - \boldsymbol{B}^{\mathrm{phy}}\cdot\delta\boldsymbol{\varphi}\right]\mathrm{d}V,\\
&= \int_{\mathscr{B}} \left[\boldsymbol{\sigma}^t : \nabla_x\delta\boldsymbol{\varphi} - \boldsymbol{b}^{\mathrm{phy}}\cdot\delta\boldsymbol{\varphi}\right]\mathrm{d}v.
\end{aligned} \tag{30}$$

Alternatively, as an exercise and for comparison with the corresponding derivations of the inverse motion viewpoint, we may write for the physical variation of the stored energy contribution to the total bulk potential energy

$$\begin{aligned}
\delta_X \int_{\mathscr{B}_0} \mathscr{W}_0\,\mathrm{d}V &= \int_{\mathscr{B}_0} \delta_X[J\mathscr{W}]\,\mathrm{d}V,\\
&= \int_{\mathscr{B}_0} [\mathscr{W}\delta_X J + J\delta_X\mathscr{W}]\,\mathrm{d}V,\\
&= \int_{\mathscr{B}} \left[\mathscr{W}\,\mathrm{Div}\,\delta\boldsymbol{\varphi} + \partial_{\boldsymbol{f}}\mathscr{W} : \delta_X\boldsymbol{f}\right]\mathrm{d}v,\\
&= \int_{\mathscr{B}} [\mathscr{W}\boldsymbol{I} - \boldsymbol{f}^t\cdot\partial_{\boldsymbol{f}}\mathscr{W}] : \nabla_x\delta\boldsymbol{\varphi}\,\mathrm{d}v,\\
&= \int_{\mathscr{B}} \boldsymbol{\sigma}^t : \nabla_x\delta\boldsymbol{\varphi}\,\mathrm{d}v.
\end{aligned} \tag{31}$$

Thus, in conclusion, identifying $\delta\boldsymbol{\varphi}$ with the virtual physical displacement $\boldsymbol{w}$, the contribution S^{sur} denotes the physical variation of the total bulk potential energy due to its complete dependence on the position in physical space, whereas the contributions S^{int} and S^{vol} denote the physical variations of the total bulk potential energy due to its implicit and explicit dependence on the position in physical space.

5.1.2. Balance of momentum

Recall that we recover the quasi-static equilibrium of physical forces in Eq. (2) if we select arbitrary uniform virtual physical displacements $\boldsymbol{w} = \boldsymbol{\theta}$ for the evaluation of Eq. (29)

$$\boldsymbol{\theta} \cdot \left[\int_{\partial \mathscr{B}} \boldsymbol{\sigma}^t \cdot \boldsymbol{n} \,\mathrm{d}a + \int_{\mathscr{B}} \boldsymbol{b}^{\mathrm{phy}} \,\mathrm{d}v \right] = 0 \quad \forall \boldsymbol{\theta} \in \mathbb{E}^3. \tag{32}$$

5.1.3. Balance of vectorial moment of momentum

Likewise, the quasi-static equilibrium of vectorial moment of physical forces in Eq. (4) is recovered if we select arbitrary rotational virtual physical displacements $\boldsymbol{w} = \boldsymbol{\theta} \times \boldsymbol{r}$ for the evaluation of Eq. (29)

$$\boldsymbol{\theta} \cdot \left[\int_{\partial \mathscr{B}} \boldsymbol{r} \times \boldsymbol{\sigma}^t \cdot \boldsymbol{n} \,\mathrm{d}a + \int_{\mathscr{B}} \left[\boldsymbol{r} \times \boldsymbol{b}^{\mathrm{phy}} + 2\,\mathrm{axl}\; \boldsymbol{\sigma} \right] \mathrm{d}v \right] = 0 \quad \forall \boldsymbol{\theta} \in \mathbb{E}^3. \tag{33}$$

5.1.4. Balance of scalar moment of momentum

Moreover, the quasi-static equilibrium of scalar moment of physical forces in Eq. (6) is recovered if we select arbitrary self similar virtual physical displacements $\boldsymbol{w} = \theta \boldsymbol{r}$ for the evaluation of Eq. (29)

$$\theta \left[\int_{\partial \mathscr{B}} \boldsymbol{r} \cdot \boldsymbol{\sigma}^t \cdot \boldsymbol{n} \,\mathrm{d}a + \int_{\mathscr{B}} \left[\boldsymbol{r} \cdot \boldsymbol{b}^{\mathrm{phy}} + 3\,\mathrm{prs}\; \boldsymbol{\sigma} \right] \mathrm{d}v \right] = 0 \quad \forall \theta \in \mathbb{R}. \tag{34}$$

5.1.5. Balance of dyadic moment of momentum

Finally, the quasi-static equilibrium of dyadic moment of physical forces in Eq. (6) is recovered if we select arbitrary virtual physical displacements $\boldsymbol{w} = \boldsymbol{\theta} \cdot \boldsymbol{r}$ for the evaluation of Eq. (29)

$$\boldsymbol{\theta} : \left[\int_{\partial \mathscr{B}} \boldsymbol{r} \otimes \boldsymbol{\sigma}^t \cdot \boldsymbol{n} \,\mathrm{d}a + \int_{\mathscr{B}} \left[\boldsymbol{r} \otimes \boldsymbol{b}^{\mathrm{phy}} + \mathrm{neg}\; \boldsymbol{\sigma} \right] \mathrm{d}v \right] = 0 \quad \forall \boldsymbol{\theta} \in \mathbb{E}^{3\times 3}. \tag{35}$$

5.2. Material quasi-static balance equations

Secondly, the pointwise or strong statement in Eq. (25) for the solution of the inverse problem is tested by a test function (virtual material displacement) $\boldsymbol{W}$ under the necessary smoothness and boundary assumptions to render

$$\underbrace{\int_{\partial \mathscr{B}_0} \boldsymbol{W} \cdot \boldsymbol{M}^t \cdot \boldsymbol{N} \,\mathrm{d}A}_{M^{\mathrm{sur}}} = \underbrace{\int_{\mathscr{B}_0} \nabla_X \boldsymbol{W} : \boldsymbol{M}^t \,\mathrm{d}V}_{M^{\mathrm{int}}} - \underbrace{\int_{\mathscr{B}_0} \boldsymbol{W} \cdot \boldsymbol{B}^{\mathrm{mat}} \,\mathrm{d}V}_{M^{\mathrm{vol}}} \quad \forall \boldsymbol{W}. \tag{36}$$

Remarkably, this variational form of the quasi-static balance of pseudomomentum has already been derived along a different line of arguments by Hill (1986). The implications of this variational statement, in particular the energetic interpretation of M^{sur}, M^{int} and M^{vol}, are discussed in the sequel.

5.2.1. Energetic interpretation

Accordingly, the different energetic terms in Eq. (36) may be interpreted by considering the material variation at fixed $\boldsymbol{x}$ of the total bulk potential energy, i.e. the quasi-static action integral

$$\begin{aligned}\delta_x\int_{\mathscr{B}}\pi\,\mathrm{d}v &= \int_{\mathscr{B}}\left[\delta_x\mathscr{W}-\boldsymbol{b}^{\mathrm{phy}}\cdot\delta_x\boldsymbol{\phi}^{-1}\right]\mathrm{d}v,\\ &= \int_{\mathscr{B}}\left[\partial_f\mathscr{W}:\nabla_x\delta\boldsymbol{\phi}+\left[\partial_{\boldsymbol{X}}\mathscr{W}+\boldsymbol{b}^{\mathrm{phy}}\cdot\boldsymbol{F}\right]\cdot\delta\boldsymbol{\phi}\right]\mathrm{d}v,\\ &= \int_{\mathscr{B}}\left[\boldsymbol{m}^t:\nabla_x\delta\boldsymbol{\phi}-\boldsymbol{b}^{\mathrm{mat}}\cdot\delta\boldsymbol{\phi}\right]\mathrm{d}v,\\ &= \int_{\mathscr{B}_0}\left[\boldsymbol{M}^t:\nabla_X\delta\boldsymbol{\phi}-\boldsymbol{B}^{\mathrm{mat}}\cdot\delta\boldsymbol{\phi}\right]\mathrm{d}V.\end{aligned}\tag{37}$$

Please note the extra term in Eq. (37) (line 2) due to the explicit dependence of $\mathscr{W}$ on $\boldsymbol{X}$ as compared to the result of the physical variation in Eq. (30) (line 2). Alternatively, again for comparison with the corresponding derivations of the direct motion viewpoint, we may write for the material variation of the stored energy contribution to the total bulk potential energy

$$\begin{aligned}\delta_x\int_{\mathscr{B}}\mathscr{W}\,\mathrm{d}v &= \int_{\mathscr{B}}\delta_x[j\mathscr{W}_0]\,\mathrm{d}v,\\ &= \int_{\mathscr{B}}[\mathscr{W}_0\delta_x j+j\delta_x\mathscr{W}_0]\,\mathrm{d}v,\\ &= \int_{\mathscr{B}_0}[\mathscr{W}_0\mathrm{Div}\,\delta\boldsymbol{\phi}+\partial_{\boldsymbol{F}}\mathscr{W}_0:\delta_x\boldsymbol{F}+\partial_{\boldsymbol{X}}\mathscr{W}_0\cdot\delta\boldsymbol{\phi}]\,\mathrm{d}V,\\ &= \int_{\mathscr{B}_0}[[\mathscr{W}_0\boldsymbol{I}-\boldsymbol{F}^t\cdot\partial_{\boldsymbol{F}}\mathscr{W}_0]:\nabla_X\delta\boldsymbol{\phi}+\partial_{\boldsymbol{X}}\mathscr{W}_0\cdot\delta\boldsymbol{\phi}]\,\mathrm{d}V,\\ &= \int_{\mathscr{B}_0}[\boldsymbol{M}^t:\nabla_X\delta\boldsymbol{\phi}+\partial_{\boldsymbol{X}}\mathscr{W}_0\cdot\delta\boldsymbol{\phi}]\,\mathrm{d}V.\end{aligned}\tag{38}$$

Note again the extra term in Eq. (38) (third to fifth lines) due to the explicit dependence of $\mathscr{W}$ on $\boldsymbol{X}$ as compared to the result of the physical variation in Eq. (31) (third to fifth lines). Thus, in conclusion, identifying $\delta\boldsymbol{\phi}$ with the virtual material displacement $\boldsymbol{W}$, the contribution M^{sur} denotes the material variation of the total bulk potential energy due to its complete dependence on the position in material space, whereas the contributions M^{int} and M^{vol} denote the material variations of the total bulk potential energy due to its implicit and explicit dependence on the position in material space.

5.2.2. Balance of momentum

Accordingly, observe that we recover the quasi-static equilibrium of material forces in Eq. (10) if we select arbitrary uniform virtual material displacements $\boldsymbol{W}=\boldsymbol{\Theta}$ for the evaluation of Eq. (36)

$$\boldsymbol{\Theta}\cdot\left[\int_{\partial\mathscr{B}_0}\boldsymbol{M}^t\cdot\boldsymbol{N}\,\mathrm{d}A+\int_{\mathscr{B}_0}\boldsymbol{B}^{\mathrm{mat}}\,\mathrm{d}V\right]=0\quad\forall\boldsymbol{\Theta}\in\mathbb{M}^3.\tag{39}$$

5.2.3. Balance of vectorial moment of momentum

Likewise, the quasi-static equilibrium of vectorial moment of material forces in Eq. (12) is recovered if we select arbitrary rotational virtual material displacements $\boldsymbol{W}=\boldsymbol{\Theta}\times\boldsymbol{R}$ for the evaluation of Eq. (36)

$$\boldsymbol{\Theta}\cdot\left[\int_{\partial\mathscr{B}_0}\boldsymbol{R}\times\boldsymbol{M}^t\cdot\boldsymbol{N}\,\mathrm{d}A+\int_{\mathscr{B}_0}[\boldsymbol{R}\times\boldsymbol{B}^{\mathrm{mat}}+2\mathrm{Axl}\;\boldsymbol{M}]\,\mathrm{d}V\right]=0\quad\forall\boldsymbol{\Theta}\in\mathbb{M}^3.\tag{40}$$

5.2.4. Balance of scalar moment of momentum

Moreover, the quasi-static equilibrium of scalar moment of material forces in Eq. (14) is recovered if we select arbitrary self similar virtual material displacements $\boldsymbol{W} = \Theta \boldsymbol{R}$ for the evaluation of Eq. (36)

$$\Theta\left[\int_{\partial\mathscr{B}_0} \boldsymbol{R}\cdot\boldsymbol{M}^t\cdot\boldsymbol{N}\,\mathrm{d}A + \int_{\mathscr{B}_0}[\boldsymbol{R}\cdot\boldsymbol{B}^{\mathrm{mat}} + 3\mathrm{prs}\,\boldsymbol{M}]\,\mathrm{d}V\right] = 0 \quad \forall\Theta\in\mathbb{R}. \tag{41}$$

5.2.5. Balance of Dyadic moment of momentum

Finally, the quasi-static equilibrium of dyadic moment of material forces in Eq. (16) is recovered if we select arbitrary virtual material displacements $\boldsymbol{W} = \boldsymbol{\Theta}\cdot\boldsymbol{R}$ for the evaluation of Eq. (36)

$$\boldsymbol{\Theta} : \left[\int_{\partial\mathscr{B}_0} \boldsymbol{R}\otimes\boldsymbol{M}^t\cdot\boldsymbol{N}\,\mathrm{d}A + \int_{\mathscr{B}_0}[\boldsymbol{R}\otimes\boldsymbol{B}^{\mathrm{mat}} + \mathrm{Neg}\,\boldsymbol{M}]\,\mathrm{d}V\right] = 0 \quad \forall\boldsymbol{\Theta}\in\mathbb{M}^{3\times3}. \tag{42}$$

Remark 5.1. *It is illuminating to note that the two variational formulations in Eqs. (29) and (36) are connected by setting* $\boldsymbol{w} = -\boldsymbol{W}\cdot\boldsymbol{F}^t$ *for the relation between the physical and material virtual displacements, compare also with the relation between the physical and material variations in Section 2. Accordingly, we find*

$$-\int_{\mathscr{B}_0} \boldsymbol{w}\cdot\mathrm{Div}\,\boldsymbol{\Sigma}^t\,\mathrm{d}V = \int_{\mathscr{B}_0}\boldsymbol{W}\cdot\boldsymbol{F}^t\cdot\mathrm{Div}\,\boldsymbol{\Sigma}^t\,\mathrm{d}V = -\int_{\mathscr{B}_0}\boldsymbol{W}\cdot\mathrm{Div}\,\boldsymbol{M}^t\,\mathrm{d}V + \int_{\mathscr{B}_0}\boldsymbol{W}\cdot\partial_{\boldsymbol{X}}\mathscr{W}_0\,\mathrm{d}V.$$

Taking into account that $-\mathrm{Div}\,\boldsymbol{\Sigma}^t = \boldsymbol{B}^{\mathrm{phy}}$, *applying integration by parts and finally the Gauss theorem renders the statements in Eqs. (29) and (36).*

6. Quasi-static path integrals

As a motivation for the application of material forces we shed a new light on the well-known quasi-static J- L- and M-integral in the context of hyperelastostatic fracture mechanics. Thereby, since the present derivations rely entirely on elementary equilibrium considerations in the material space they differ essentially from well-known expositions in the literature and therefore seem to be new.

6.1. Equilibrium of material forces

To this end, we firstly consider an arbitrary subdomain $\mathscr{V}_0$ of the reference configuration $\mathscr{B}_0$ in Fig. 7. Thereby, the boundary $\partial\mathscr{V}_0$ is assumed to be decomposed into a regular and a singular part $\partial\mathscr{V}_0 = \partial\mathscr{V}_0^{\mathrm{r}}\cup\partial\mathscr{V}_0^{\mathrm{s}}$ with $\emptyset = \partial\mathscr{V}_0^{\mathrm{r}}\cap\partial\mathscr{V}_0^{\mathrm{s}}$. Here the singular part of $\partial\mathscr{V}_0$ denotes a crack tip.

For an inhomogeneous material with nonvanishing distributed material volume forces $\boldsymbol{B}^{\mathrm{mat}}\neq\boldsymbol{0}$ within $\mathscr{V}_0$ Eq. (10) renders the following relation between the material surface and volume forces:

$$\int_{\partial\mathscr{V}_0}\boldsymbol{M}^t\cdot\boldsymbol{N}\,\mathrm{d}A = -\int_{\mathscr{V}_0}\boldsymbol{B}^{\mathrm{mat}}\,\mathrm{d}V. \tag{43}$$

Taking the decomposition of the boundary $\partial\mathscr{V}_0$ into a regular and a singular boundary into account, the resulting material force acting on the singular boundary is given by

$$\boldsymbol{F}^{\mathrm{mat,sng}} := \int_{\partial\mathscr{V}_0^{\mathrm{s}}}\boldsymbol{M}^t\cdot\boldsymbol{N}\,\mathrm{d}A = -\int_{\partial\mathscr{V}_0^{\mathrm{r}}}\boldsymbol{M}^t\cdot\boldsymbol{N}\,\mathrm{d}A - \int_{\mathscr{V}_0}\boldsymbol{B}^{\mathrm{mat}}\,\mathrm{d}V. \tag{44}$$

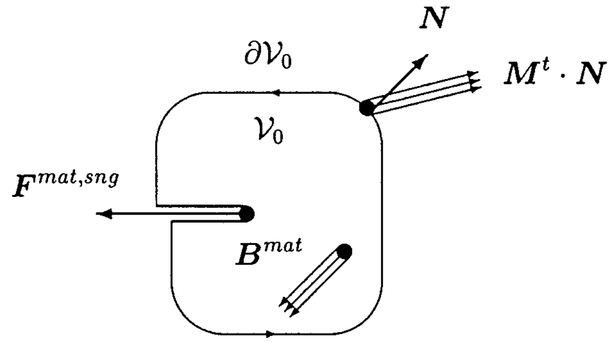

Fig. 7. Arbitrary subdomain with regular and singular part of its boundary: a material single force acts on the singular part of the boundary.

Please note that this material force coincides with the (vectorial) J-integral as originally proposed by Rice (1968) modulo a change of sign which stems from the integration along the regular part instead along the singular part of ∂V_0, see also Remarks 6.1 and 6.4.

$$\boldsymbol{J} = \lim_{\partial\mathscr{V}_0^{\mathrm{r}} \to 0} \int_{\partial\mathscr{V}_0^{\mathrm{r}}} \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A. \tag{45}$$

6.2. Equilibrium of vectorial moment of material forces

Next, we consider an arbitrary subdomain $\mathscr{V}_0$ of the reference configuration $\mathscr{B}_0$ with an embedded crack, (Fig. 8). To this end the boundary $\partial\mathscr{V}_0$ is assumed to be decomposed into an external and an internal part $\partial\mathscr{V}_0 = \partial\mathscr{V}_0^{\mathrm{e}} \cup \partial\mathscr{V}_0^{\mathrm{i}}$ with $\emptyset = \partial\mathscr{V}_0^{\mathrm{e}} \cap \partial\mathscr{V}_0^{\mathrm{i}}$. Here the internal part of $\partial\mathscr{V}_0$ encircles the crack.

For an inhomogeneous material with nonvanishing distributed material volume forces $\boldsymbol{B}^{\mathrm{mat}} \neq \boldsymbol{0}$ within $\mathscr{V}_0$, Eq. (12) renders the following relation between the vectorial moment of material surface and volume forces:

$$\int_{\partial\mathscr{V}_0} \boldsymbol{R} \times \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A = -\int_{\mathscr{V}_0} [\boldsymbol{R} \times \boldsymbol{B}^{\mathrm{mat}} + 2\mathrm{Axl}\ \boldsymbol{M}]\,\mathrm{d}V. \tag{46}$$

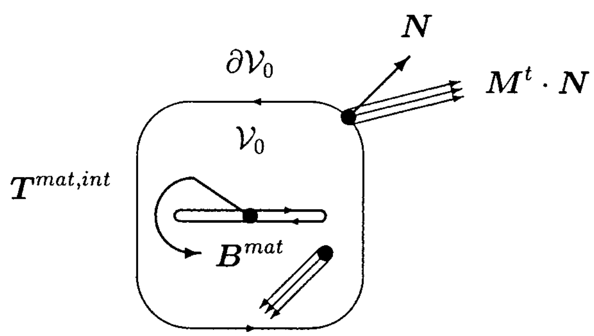

Fig. 8. Arbitrary subdomain with external and internal part of its boundary: a resulting material torque acts on the internal part of the boundary.

Taking the decomposition of the boundary $\partial\mathscr{V}_0$ into an external and an internal boundary into account, the resulting vectorial moment of material forces, i.e. the material torque acting on the internal boundary is given by

$$\boldsymbol{T}^{\text{mat,int}} := \int_{\partial\mathscr{V}_0^{\text{i}}} \boldsymbol{R} \times \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A = -\int_{\partial\mathscr{V}_0^{\text{e}}} \boldsymbol{R} \times \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A - \int_{\mathscr{V}_0} [\boldsymbol{R} \times \boldsymbol{B}^{\text{mat}} + 2\text{Axl}\ \boldsymbol{M}]\,\mathrm{d}V. \tag{47}$$

Please note that this material torque essentially coincides with the vectorial L-integral as proposed e.g. by Günther (1962), Knowles and Sternberg (1972) and interpreted later on by Budiansky and Rice (1973) for the geometrically linear case modulo a change of sign which stems from the integration along the external part instead along the internal part of ∂V_0, see also Remarks 6.1 and 6.4

$$\boldsymbol{L} = \lim_{\partial\mathscr{V}_0^{\text{e}} \to \partial\mathscr{V}_0^{\text{i}}} \int_{\partial\mathscr{V}_0^{\text{e}}} \boldsymbol{R} \times \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A. \tag{48}$$

6.3. Equilibrium of scalar moment of material forces

Then, we consider again the same subdomain $\mathscr{V}_0$ with embedded crack (Fig. 9). For an inhomogeneous material with nonvanishing distributed material volume forces $\boldsymbol{B}^{\text{mat}} \neq \boldsymbol{0}$ within $\mathscr{V}_0$, Eq. (14) renders the following relation between the scalar moments of material surface and volume forces:

$$\int_{\partial\mathscr{V}_0} \boldsymbol{R} \cdot \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A = -\int_{\mathscr{V}_0} [\boldsymbol{R} \cdot \boldsymbol{B}^{\text{mat}} + 3\text{prs}\ \boldsymbol{M}]\,\mathrm{d}V. \tag{49}$$

Taking the decomposition of the boundary $\partial\mathscr{V}_0$ into an external and an internal boundary into account, the resulting scalar moment of material forces, or rather the material virial, contributed by the internal boundary is given by

$$V^{\text{mat,int}} := \int_{\partial\mathscr{V}_0^{\text{i}}} \boldsymbol{R} \cdot \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A = -\int_{\partial\mathscr{V}_0^{\text{e}}} \boldsymbol{R} \cdot \boldsymbol{M}^t \cdot \boldsymbol{N}\,\mathrm{d}A - \int_{\mathscr{V}_0} [\boldsymbol{R} \cdot \boldsymbol{B}^{\text{mat}} + 3\text{prs}\ \boldsymbol{M}]\,\mathrm{d}V. \tag{50}$$

Please note that this material virial is closely related to the scalar M-integral as proposed for the geometrically linear case e.g. by Günther (1962), Knowles and Sternberg (1972), see also Remarks 6.2 and 6.4.

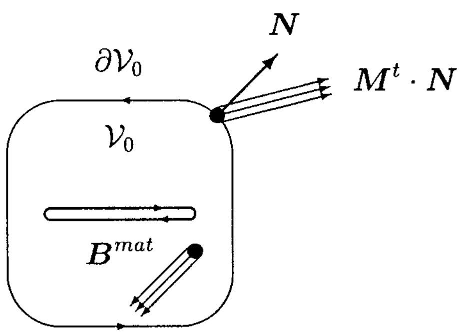

Fig. 9. Arbitrary subdomain with external and internal parts of its boundary: a resulting material virial acts on the internal parts of the boundary.

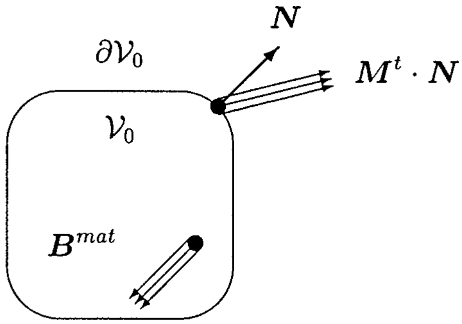

Fig. 10. Arbitrary subdomain with material traction on the boundary and material volume force in the bulk.

6.4. Equilibrium of dyadic moment of material forces

Finally, we consider an arbitrary subdomain $\mathscr{V}_0$ of the reference configuration $\mathscr{B}_0$ in Fig. 10. Then, for an inhomogeneous material with nonvanishing distributed material volume forces $\boldsymbol{B}^{\text{mat}} \neq \boldsymbol{0}$ within $\mathscr{V}_0$, Eq. (16) renders the following averaging relation between the dyadic moment of material surface and volume forces

$$\int_{\partial\mathscr{V}_0} \boldsymbol{R} \otimes \boldsymbol{M}^t \cdot \boldsymbol{N}\, \mathrm{d}A = -\int_{\mathscr{V}_0} [\boldsymbol{R} \otimes \boldsymbol{B}^{\text{mat}} + \text{Neg}\ \boldsymbol{M}]\, \mathrm{d}V. \tag{51}$$

As an application, Eq. (51) might in particular be used to compute the average Eshelby stress in $\mathscr{V}_0$ from the boundary data on $\partial\mathscr{V}_0$ for vanishing material volume forces. For further discussion refer to Remark 6.3.

Remark 6.1. *Clearly, issues of path dependence of the J- and L-integrals can now easily be discussed based on straightforward material equilibrium considerations. For example, for unloaded straight crack surfaces, no physical volume forces and homogeneous material of the component of* $\boldsymbol{J}$ *parallel to the crack surfaces proves to be path independent. Likewise, for the case of no physical volume forces and homogeneous, isotropic material* $\boldsymbol{L}$ *proves as well to be path independent.*

Remark 6.2. *Due to the present definition of* $V^{\text{mat,int}}$ *the integral in Eq. (50) is not readily path independent even for the assumptions of vanishing physical volume forces and homogeneous material. Nevertheless, for hyperelastic materials with a stored energy function* $\mathscr{W}_0$ *per unit volume in* $\mathscr{B}_0$ *which is homogeneous of degree k in* $\boldsymbol{F}$, *i.e.* $\mathscr{W}_0(\varepsilon\boldsymbol{F}) = \varepsilon^k \mathscr{W}_0(\boldsymbol{F})$ *and* $\partial_{\boldsymbol{F}}\mathscr{W}_0 : \boldsymbol{F} = k\mathscr{W}_0$, *we may transform the remaining volume integral in Eq. (50) into a surface integral by noting that* $k\mathscr{W}_0 = -3\text{prs}\ (J\boldsymbol{\sigma})$, *thus with the result in Remark 4.3 we have*

$$-\int_{\mathscr{V}_0} 3\text{prs}\ \boldsymbol{M}\, \mathrm{d}V = \int_{\mathscr{V}_0} [3\mathscr{W}_0 + 3\text{prs}\ (J\boldsymbol{\sigma})]\, \mathrm{d}V = \frac{k-3}{k}\int_{\mathscr{V}_0} 3\text{prs}\ (J\boldsymbol{\sigma})\, \mathrm{d}V = \frac{3-k}{k}\int_{\partial\mathscr{V}_0} \boldsymbol{r} \cdot \boldsymbol{\Sigma}^t \cdot \boldsymbol{N}\, \mathrm{d}A.$$

Under the aformentioned conditions, we may then define the path independent M-integral via

$$M = -V^{\text{mat,int}} - \frac{k-3}{k}\int_{\partial\mathscr{V}_0^{\text{i}}} \boldsymbol{r} \cdot \boldsymbol{\Sigma}^t \cdot \boldsymbol{N}\, \mathrm{d}A = \lim_{\partial\mathscr{V}_0^{\text{e}} \to \partial\mathscr{V}_0^{\text{i}}} \int_{\partial\mathscr{V}_0^{\text{e}}} \left[\boldsymbol{R} \cdot \boldsymbol{M}^t + \frac{k-3}{k}\boldsymbol{r} \cdot \boldsymbol{\Sigma}^t\right] \cdot \boldsymbol{N}\, \mathrm{d}A.$$

Remark 6.3. *For vanishing material volume forces and isotropic material, the resulting symmetry of* $\boldsymbol{M}$ *in Eq. (51) induces a path integral*

$$\int_{\partial\mathscr{V}_0} [\boldsymbol{R} \otimes \boldsymbol{M}^t \cdot \boldsymbol{N}]^{\mathrm{Skw}} \, \mathrm{d}A = \boldsymbol{0}.$$

This particular version of the symmetry condition for $\boldsymbol{M}$ *in the case of isotropy was pointed out by* Chadwick (1975) *and* Hill (1986). *Path independence of the above integral is trivially assured, but of course the numerical result as such is of no particular practical value. However, the axial vector of the integral* $[\boldsymbol{R} \otimes \boldsymbol{M}^t \cdot \boldsymbol{N}]^{\mathrm{Skw}}$ *is given by* $-\frac{1}{2}\boldsymbol{R} \times \boldsymbol{M}^t \cdot \boldsymbol{N}$, *thus for the particular case of homogeneous, isotropic material the above condition coincides with the equilibrium of vectorial moment of material forces in Eq. (46) and consequently the L-integral may be recovered.*

Remark 6.4. *With* $\boldsymbol{M} = \tilde{\boldsymbol{M}} - \boldsymbol{\Sigma}$ *we may rewrite the J- and L-integrals for the path independent case with unloaded crack surfaces and crack tips in a more common way as*

$$\boldsymbol{J} = \int_{\partial\mathscr{V}_0^{\mathrm{r}}} \tilde{\boldsymbol{M}}^t \cdot \boldsymbol{N} \, \mathrm{d}A \quad \text{and} \quad \boldsymbol{L} = \int_{\partial\mathscr{V}_0^{\mathrm{e}}} \left[\boldsymbol{R} \times \tilde{\boldsymbol{M}}^t + \boldsymbol{u} \times \boldsymbol{\Sigma}^t \right] \cdot \boldsymbol{N} \, \mathrm{d}A.$$

If in addition we assume alternatively $\mathscr{W}_0$ *homogeneous of degree k in* $\boldsymbol{H}$ *the M-integral for the path independent case follows in the format familiar in the geometrically linear setting as*

$$M = \int_{\partial\mathscr{V}_0^{\mathrm{e}}} \left[\boldsymbol{R} \cdot \tilde{\boldsymbol{M}}^t + \frac{k-3}{k} \boldsymbol{u} \cdot \boldsymbol{\Sigma}^t \right] \cdot \boldsymbol{N} \, \mathrm{d}A.$$

7. Conclusions

The main goal of this work was the application of the notion of material forces to quasi-static hyperelastic fracture mechanics. Thereby, once the striking duality of physical and material forces and moreover the duality of Newton type and Eshelby type stress measures along with the corresponding balance equations has been accepted, elementary equilibrium considerations in the material space render the classical *J*-, *L*- and *M*-integrals. To this end, even for the geometrically nonlinear case no assumptions on the type of singularity or considerations of the energy changes etc are necessary, thus making the present approach conceptually extremely straightforward and easy. It is in particular this feature, i.e. the possibility to operate essentially with elementary equilibrium considerations, that constitutes the main benefit of the advocated viewpoint. The interpretation of the *J*-integral as a material force is immediately complemented by the *L*- and *M*-integrals being interpreted as first order moments of material forces. Besides this main thrust, a number of interesting aspects have been addressed. Symmetry conditions for the Newton type and Eshelby type stress measures with respect to the standard euclidian and appropriate strain metrics have been associated with the requirements of physical objectivity, i.e. frame indifference, and material objectivity, i.e. material isotropy, respectively. Equilibrium conditions and balance equations associated with translations in physical and material space have consequently been extended to all possibilities for their first order moments, whereby conditions for the independence of these additional equations have been discussed wherever appropriate. Finally, as a prerequisite for discretization methods to be described in Part II of this work, the weak forms of the quasi-static balance equations have been investigated and the energetic interpretation of the separate terms involved has been discussed.

In summary, it is believed that this contribution clarifies and unifies issues pertaining to the direct and inverse motion viewpoint of continuum mechanics with particular application to hyperelastostatic fracture mechanics. Moreover, it establishes the necessary basis for the forthcoming Part II of this work which will focus on the computational setting associated to the theoretical consideration advocated in the present Part I.

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